13

Math : Differentiation

► Dr.Wattana Toutip - Department of Mathematics – Khon Kaen University

 ◙ wattou@kku.ac.th ◙ http://home.kku.ac.th/wattou

13. Differentiation

If y is a function of x, the rate of change of y with respect to x is called the derivative or the differential coefficient of y. It represents the gradient of the graph of y against x.

The derivative of y is written $\frac{dy}{dx}$.

If the function is of y is of the form $y = f(x)$,then the derivative is sometimes written as

$\frac{dy}{dx} = f'(x)$.

The process of going from y to $\frac{dy}{dx}$ is called differentiation.

The result of differentiating twice, i.e. of differentiating the derivative, is the second derivative.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

13.1 Differentiation of powers of x

Let y be a power of x,(even a negative or a fractional power of x). Differentiate by multiplying by the power and subtracting 1 from the power.

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$.

If a function is multiplied by a constant, then the derivative is also multiplied by the constant.

If $y = ax^n$, then $\frac{dy}{dx} = anx^{n-1}$

If two functions are asses, then their derivatives are also added.

If $y = ax^n + bx^m$, then $\frac{dy}{dx} = anx^{n-1} + bmx^{m-1}$

Two special cases are as follows:

If y is a constant , then $\frac{dy}{dx} = 0$.

If y is the straight line $y = mx + c$, then $\frac{dy}{dx} = m$.

13.1.1 Examples

1. Differentiate $y = 3x^2 - \frac{2}{x} + \sqrt{x}$.

   Solution

   Write y in terms of powers of x.

   $y = 3x^2 - 2x^{-1} + x^{\frac{1}{2}}$

   Apply the formulae above , using $n = 2$, $n = -1$ and $n = \frac{1}{2}$ respectively.

   $\frac{dy}{dx} = 3(2)x^{2-1} - 2(-1)x^{-1-1} + \frac{1}{2}x^{\frac{1}{2}-1}$

   $\frac{dy}{dx} = 6x + 2x^{-2} + \frac{1}{2}x^{-\frac{1}{2}}$

2. Differentiate $y = \frac{(x+3)(x-4)}{x^2}$

   Solution

   Before differentiating it is necessary to multiply out the expression and divide by x^2.

   $y = \frac{x^2 - x - 12}{x^2} = 1 - x^{-1} - 12x^{-2}$

   Now use the formulae above.

   $\frac{dy}{dx} = -(-1)x^2 - 12 \times (-2) \times x^{-3}$

   $= x^{-2} + 24x^{-3}$

   $= \frac{1}{x^2} + \frac{24}{x^3}$

13.1.2 Exercises

1. Differentiate the following :

   (a) $y = x^3$

   (b) $y = 7x^5$

   (c) $y = 3x^2 + 2x$

   (d) $y = 2 - 2x + 4x^5$

   (e) $y = 2x^{-3}$

   (f) $y = 2x - 3x^{-4}$

   (g) $y = 4x^{1/2}$

   (h) $y = 3x^{-3} + 2x^{-1/2} + 17$

(i) $y=\frac{7}{x^3}$

(j) $y=\frac{2}{x}-\frac{8}{x^3}$

(k) $y=\sqrt{x}$

(l) $y=3\sqrt{x}-\frac{7}{\sqrt{x}}$

(m) $y=2x^5-\frac{5}{x^6}+\frac{7}{x^{1/4}}$

(n) $y=2+\frac{2}{\sqrt{x}}-\frac{9}{x}$

2. Differentiate the following:

(a) $y=x(x-3)$

(b) $y=\left(x+1\right)\left(x-6\right)$

(c) $y=\left(2-3x^2\right)\left(4+x^3\right)$

(d) $y=x\left(1+\frac{1}{x}\right)\left(2+\frac{3}{x^3}\right)$

(e) $y=\left(x+2\right)^2$

(f) $y=\left(3+2x^2\right)^2$

(g) $y=\left(2+\frac{3}{x}\right)^2$

(h) $y=\left(3x-\frac{4}{x}\right)^2$

(i) $y=x^{1/2}\left(x-4\right)$

(j) $y=\left(x^{1/2}+1\right)\left(x^{1/2}-2\right)$

(k) $y=\frac{x+17}{x^4}$

(l) $y=\frac{2x^2-3x+1}{x}$

(m) $y=\frac{x+8}{\sqrt{x}}$

(n) $y=\frac{\sqrt[3]{x}-\sqrt[4]{x}+3}{\sqrt{x}}$

3. Find the second derivatives of the following:

(a) $y=x^3$

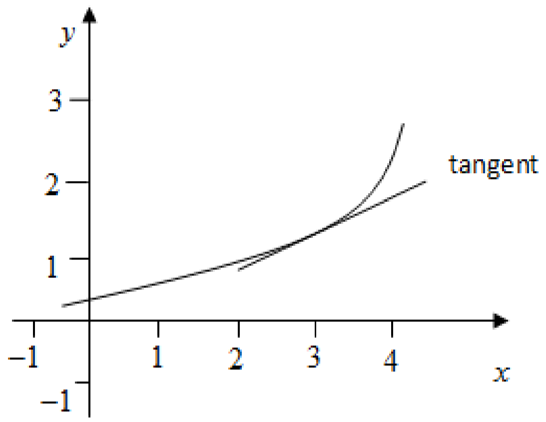
(b) $y=2x^2-3x+1$

(c) $y=(x^3+1)(x-4)$

(d) $y=3\sqrt{x}-2x^{-2}$

13.2 Tangent and normal

The derivative of a function at a point is the gradient of the tangent if the function at that point. (Fig 13.1)



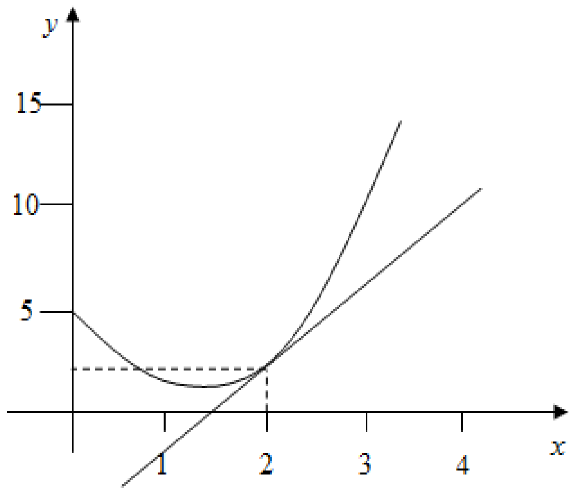


Fig 13.1

Fig 13.2

13.2 Examples

1. Find the equation of the tangent to the curve $y=3x^2-7x+5$ at the point $(2,3)$. (Fig 13.2)
   Solution
   Differentiate to fine that $\dfrac{dy}{dx}=6x-7$. When $x=2$, the gradient of the tangent is $6\times 2-7=5$.
   The tangent is of the form $y=5x+c$. The tangent goes through the point $(2,3)$,hence $c=3-5\times 2=-7$.
   The tangent is $y=5x-7$
2. At what points of the curve $y=4x^3-12.5x$ is the normal parallel to $y=2x+4$?
   Solution
   The gradient of the line $y=2x+4$ is 2.
   The gradient of the curve $y=4x^3-12.5x$ is $\dfrac{dy}{dx}=12x^2-12.5$.
   If these are at right angles, then: $2(12x^3-12.5)=-1$
   $24x^2=-1$
   $x=1$ or -1.

The points are $(1,-8.5)$ and $(-1,8.5)$

13.2.2 Exercises

1. Find the equations of the tangents to the following curves at the points given.
   (a) $y = x^2$ at $(2,4)$
   (b) $y = x^3 - 2x$ at $(1,-1)$
   (c) $y = 3x^2 - 7x + 2$ at $(-1,12)$
   (d) $y = 1 - \frac{2}{x}$ at $\left(4, \frac{1}{2}\right)$
2. Find the normals to the curves in Question 1 at the same points.
3. Where is the tangent to $y = 3x^2 + 7x - 4$ parallel to the x-axis .
4. Find the points where the tangent to $y = x^3 - x$ are parallel to the x-axis.
5. Show that the tangent to the curve $y = x^3 + 2x$ are never parallel to the x-axis.
6. The line $y = x + 7$ is a tangent to the curve $y = x^2 + a$, where a is a constant. Find the value of a.

13.3 Maxima and minima

A flat point of a curve is a stationary point. At a stationary point, $dy/dx = 0$.
A point of a curve where it changes from convex to concave or vice is a point of inflection.
At a point of inflection, $\frac{d^2y}{dx^2} = 0$.
There are three kinds of stationary point.

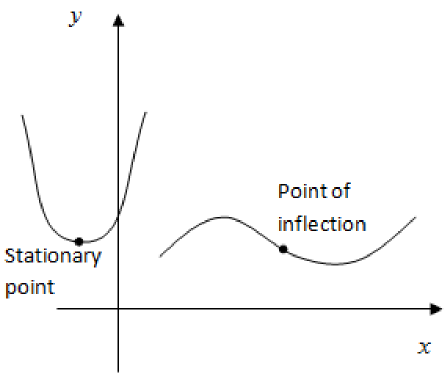


Fig 13.3

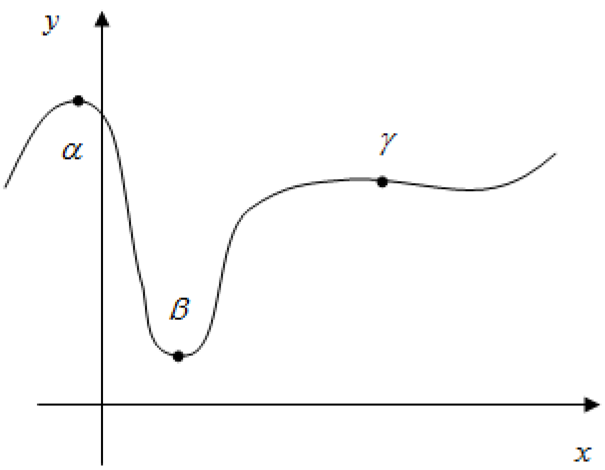


Fig 13.4

A point where y is at top of a peak is maximum (α). A point where y is at the bottom of a pit is a minimum (β). A point where the curve is flat but continues to go up or down is a point of inflection (γ).

There are three ways to distinguish between these three kinds of stationary point.

1. Find the values of y on both sides of the point.
2. Find the values of $\frac{dy}{dx}$ on both sides of the point.
3. Use the second derivative.

   If $\frac{d^2y}{dx^2} > 0$ then it is minimum.

   If $\frac{d^2y}{dx^2} < 0$ then it is a maximum.

   If the second derivative is 0, then use either 1 or 2.

13.3.1 Examples

1. Find the maxima and minima of $y = 2x^3 + 3x^2 - 12x + 4$ distinguishing between them. Sketch the graph of the function.

   Solution

   Differentiating , $\frac{dy}{dx} = 6x^2 + 6x - 12$. This is zero at a maximum or a minimum.

   $6x^2 + 6x - 12 = 0$

   $6(x-1)(x+2) = 0$

   $x = 1, y = -3$ and $x = -2, y = 24$.

   To find out whether $(1,-3)$ is a maximum or a minimum, find the value of y on either side of the point.

   For $x = 0.9, y = -2.9$. For $x = 1.1, y = -2.91$

   $(1,-3)$ is a minimum. Similarly, $(-2,24)$ is a maximum.

   Sketch the graph as shown, with a peak at $(-2,24)$ and a pit at $(1,-3)$.

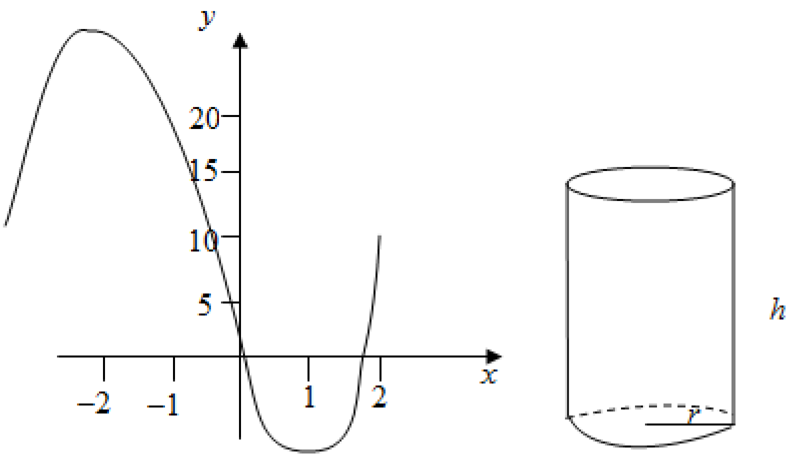


Fig 13.6

2. Thin metal is used make cylindrical cans which are to hold $1,000cm^3$ of soup. What should be the cans if they are to use as little metal as possible? (Fig 13.6)

Solution

Let the height be h cm and the base radius r cm. The volume is $1,000cm^3$, hence $\pi r^2 h = 1,000$.

The surface area is A cm^2, where $A = 2\pi r^2 + 2\pi rh$. Write h in terms of r so that A is in terms of r only.

$h = \frac{1,000}{\pi r^2}$, so $A = 2\pi r^2 + \frac{2,000}{r}$

Differentiate to find where A is a minimum.

$\frac{dA}{dr} = 4\pi r - \frac{2,000}{r^2}$.

At a minimum, $\frac{dA}{dr} = 0$

$4\pi r = \frac{2,000}{r^2}$

$r^3 = \frac{2,000}{4\pi}$.

The best radius is 5.42 cm.

13.3.2 Exercises

1. Find the coordinates of the maxima and minima of the following, distinguishing between them.

(a) $y = x^2 - 2x$

(b) $y = 2 + 4x - x^2$

(c) $y = x^3 - 3x$

(d) $y = 2x^3 - 15x^2 + 36x - 4$

(e) $y=x+\frac{1}{x}$

(f) $y=x+\frac{4}{x^2}$

2. Find the points of inflection of the functions of Question 1.
3. The function $y=x^2-2x+a$ has a minimum of 5. Find the value of a.
4. The function $y=ax^2+bx$ has a maximum at $(2,5)$. Find the values of a and of b.
5. A metal basket is made in the form of a cylinder with as open top. Its height is h cm. It is to contain $5,000\ cm^3$. What is the radius , if it is to consist of the smallest possible amount of metal?
6. A cardboard box is to be made in the form of a cuboid with square cross-section. Its volume must be $2,000\ cm^3$. Let $x\ cm$ be the side of the square , and z cm be the length. What value of x will use the least cardboard?
7. Repeat Question 6,if the squares at the end use a double thickness of cardboard.
8. Repeat Example 2 of 13.3.1, if the top and bottom of the can use a double thickness of metal.
9. A coboid box is to have a volume of $512\ cm^3$. The length is twice the breadth. Find the length, if the surface area is to be as small as possible.
10. A farmer has 100 m of fencing, with which to make a rectangular sheepfold. One side of the fold is already provided by a stone wall. How should he make his fold if he is to enclose as much area as possible?
11. A 100 cm length of wire is cut in two piece, which are then made into two squares. Show that the total area enclosed by the square is least when they have equal sides.
12. 100 cm of wire is cut in two pieces, one of which are then made into a square and the other into a circle. Find is made into a square and the other into a circle. Find the radius of the circle if the total area is to be as a small as possible.
13. The line through $(1,2)$ with gradient m is $y-2=m(x-1)$. Find where this line crosses the axes. Find the value of m which will make the area enclosed by the line and the two axes as small as possible.
14. If a book is sold for Lx, the number of copies sold will be $1000(10-x)$. What value of x will make the revenue from sales of the book as great as possible?
15. The cost of making a tie is $L2$. If the tie cists Lx, the number sold will be $100(25-x^2)$. What price will maximize:
    (a) The revenue
    (b) The profit

13.4 Examination questions

1. Find $\frac{dy}{dx}$ when:

   (i) $y=x^3+\frac{1}{x^3}$

   (ii) $y=x(x-1)^2$

(iii) $y = x^{3/2}$ and $x = 9$

(iv) $y = \dfrac{x^2+1}{x}$ and $x = 2$

2. A curve has equation $y = \frac{1}{3}x^3 - 4x^2 + 13x$.

   (a) Show that the tangent at the point where $x = 2$ has gradient 1 and find its equation.

   (b) Find the x-coordinate of another point where the tangent has gradient 1.

   (c) Show that there is only one tangent with gradient -3

3. The point $P(0, 2)$, lies on the curve $y = x^3 - 3x + 2$.

   The tangent and normal to the curve at P intersect the x-axis at A and B respectively. Calculate the length of AB.

4. In making closed cylindrical tins of height h cm and radius r cm the walls can be made from rectangular sheets of width h cm and length $2\pi r$ cm without waste, but stamping out each circular end of are πr^2 cm^2 requires an area $4r^2$ cm^2 and some is wasted.

   If the tins are to contain 1000 cm,

(i) Find h in term of r to five the correct volume;

(ii) Find the area of tinplate required for each tin in terms of r.

If tinplate costs 0.01 p per cm^2, find by a calculus method the radius, height and cist of the tin which will contain the required volume mist cheaply.

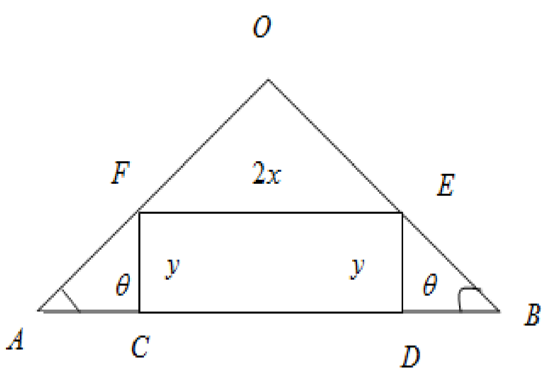
5. At which point (other than the origin) do the parabolas $y = 4ax$ and $x^2 = 4ay$ meet?

Write down the equation tangent to each parabola at this point.

If the angles which these tangents make with the positive direction of the x-axis are θ_1 and θ_2 show that $|\tan(\theta_1 - \theta_2)| = \dfrac{3}{4}$.

6. In the diagram, $CDEF$ is a rectangle with $CD = FE = 2x$ and $CF = DE = y$, and OAB is a triangle with angles at A and B each equal to θ.

   Fig13.7



(a) Show that the area, Δ, of the triangle OAB is given by

$$\Delta = \left(x + \frac{y}{t}\right)(y + tx),$$

Where $t = \tan\theta$

Deduce that, as t varies, with x and y remaining fixed, the least value of Δ is $4xy$.

(b) Given further that $OA = OB = l$, show that $y = l\sin\theta - x\tan\theta$, and that the area, S, of the rectangle $CDEF$ is given by

$S = 2[x\sin\theta - x^2 \tan\theta]$.

Deduce that ,as x varies with l and θ remaining fixed, the greatest value of S is $\frac{1}{2}l^2 \sin\theta\cos\theta$

Common errors

1. Differentiation

   There are a number of common mistakes made when differentiating . Be careful of ngative powers, of constants, of products and of ratios. Watch out for the following:

   If $y = x^5$, $\frac{dy}{dx} \neq -5x^{-4}$. It should be $-5x^6$

   If $y = x + 7$, $\frac{dy}{dx} \neq 2x + 7$ .It should be $2x$.

   If $y = x^2 \times x^3$, $\frac{dy}{dx} \neq 2x \times 3x^2$ .It should be $5x^4$.

   If $y = \frac{x^6}{x^2}$, $\frac{dy}{dx} \neq \frac{6x^5}{2x}$. Should be $4x^3$.

2. Maxima and minima

   When you asked for the maximum or minimum value, then you must give the y value, not the x value. If you are asked for the point at which it is a maximum, give both the x and the y values.

   If $\frac{d^2y}{dx^2} = 0$ at a stationary point, then do not conclude that it is a point of inflection.

   Ecamine either y or $\frac{dy}{dx}$ on both sides of the point .

   When you are solving problems about minimum area, make sure that your area is expressed in terms of only one variable. If the area is in terms of both the height and the width then it cannot be differentiated.

Solution (to exercise)
13.1.2

1.
   (a) $3x^2$
   (b) $35x^4$
   (c) $6x+2$
   (d) $-2+20x^4$
   (e) $-6x^4$
   (f) $2+12x^{-5}$
   (g) $2x^{-1/2}$
   (h) $-9x^{-4}-x^{-3/2}$
   (i) $-21x^{-4}$
   (j) $-2x^2+24x^{-4}$
   (k) $\frac{1}{2}x^{-1/2}$
   (l) $\frac{3}{2}x^{-1/2}+\frac{7}{2}x^{-3/2}$
   (m) $10x^4+30x^{-7}-\frac{7}{4}x^{-5/4}$
   (n) $-x^{-3/2}+9x^{-2}$
2.
   (a) $2x-3$
   (b) $2x-5$
   (c) $-15x^4+6x^2-24x$
   (d) $2-6x^{-3}-9x^{-4}$
   (e) $2x+4$
   (f) $24x+16x^3$
   (g) $-12x^2-18x^{-3}$
   (h) $18x-32x^{-3}$
   (i) $\frac{3}{2}x^{1/2}-2x^{-1/2}$
   (j) $1-\frac{1}{2}x^{-1/2}$
   (k) $-3x^{-4}-68x^{-5}$
   (l) $2-x^{-2}$
   (m) $\frac{1}{2}x^{-1/2}-4x^{-3/4}$
   (n) $-\frac{1}{6}x^{-7/6}+\frac{1}{4}x^{-5/4}-\frac{3}{2}x^{-3/2}$
3.

(a) $6x$
(b) 4
(c) $12x^2 - 24x$
(d) $-\frac{3}{4}x^{-3/2} - 12x^{-4}$

13.2.2

1.
(a) $y = 4x - 4$
(b) $y = x - 2$
(c) $y = -13x - 1$
(d) $y = \frac{1}{8}x$

2.
(a) $y = -\frac{1}{4}x + 4\frac{1}{2}$
(b) $y = -x$
(c) $3y = x + 157$
(d) $y = -8x + 32\frac{1}{2}$

3. $x = -\frac{2}{3}$

4. $\pm\sqrt{\frac{1}{3}}$

6. $7\frac{1}{4}$

13.3.2

1.
(a) $(1,-1)$ min
(b) $(2,6)$ max
(c) $(1,-2)$ min, $(-1,2)$ max
(d) $(2,24)$ max, $(3,23)$ min
(e) $(1,2)$ min, $(-1,-2)$ max
(f) $(2,3)$ min

2.
(a) None
(b) None
(c) $(0,0)$
(d) $(2.5, 23.5)$
(e) None
(f) None

3. 6
4. $a = -\frac{3}{4}, b = 3$
5. $11.7cm$
6. $12.6cm$
7. $10cm$
8. $4.3cm$
9. $7.69cm$
10. $25m$ by $50m$
12. $7cm$
13. $(0, 2, -m), (1-2/m, 0), m = \pm 2$
14. 5
15. 2.89
16. 3.63

==

References:

Solomon, R.C. (1997), *A Level: Mathematics* (4th Edition) , Great Britain, Hillman Printers(Frome) Ltd.

More: (in Thai)
http://home.kku.ac.th/wattou/service/m456/09.pdf