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14 Further Differentiation

14.1 The product and quotient rules

These rules enable products and quotients of functions to be differentiated.

The Product Rule. If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

The Quotient Rule. If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

14.1.1 Examples

1. Differentiate $y = \left(x^2 + 3x - 2\right)\left(4 - x^2\right)$

   Solution

   Here $u = x^2 + 3x - 2$ and $v = 4 - x^2$. Apply the Product Rule:

   $$\frac{dy}{dx} = \left(x^2 + 3x - 2\right)\left(-2x\right) + \left(4 - x^2\right)\left(2x + 3\right)$$

   $$\frac{dy}{dx} = -4x^3 - 9x^2 + 12x + 12$$

2. Differentiate $y = \frac{x}{2x^2 - 3x + 5}$

   Solution

   Here $u = x$ and $v = 2x^2 - 3x + 5$. Apply the Quotient Rule:

   $$\frac{dy}{dx} = \frac{\left(2x^2 - 3x + 5\right) \times 1 - x \times \left(4x - 3\right)}{\left(2x^2 - 3x + 5\right)^2}$$

   $$\frac{dy}{dx} = \frac{-2x^2 + 5}{\left(2x^2 - 3x + 5\right)^2}$$

14.1.2 Exercises

1. Without multiplying out the brackets, differentiate the following:

   (a) $y = x\left(x^2 + 7x - 3\right)$

(b) $y=(x+3)(2x^3+6x)$

(c) $y=x^2(3x^5-3x^2+x)$

(d) $y=(x^2-1)(x^2+1)$

2. Differentiate the following:

(a) $y=\frac{x}{x+3}$

(b) $y=\frac{x^2+7x-2}{x^2+3}$

(c) $y=\frac{3x^2}{x-17}$

(d) $y=\frac{1}{x+3}$

3. Find the equation of the tangent to the curve $y=\frac{x^2}{x-1}$ at the point $(2,4)$. Find the equation of the normal at $(3,4.5)$.

4. Find the maxima and minima of the following functions

(a) $y=\frac{x^2}{x-1}$

(b) $y=\frac{x^2+1}{x+1}$

(c) $y=\frac{1}{x^2+1}$

(d) $y=\frac{1}{x^2+4x+9}$

14.2 The chain rule

The Chain Rule enables functions of functions to be differentiated. If y is a function of z, and z is a function of x, then the derivative of y is found by the following :

$$\frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}$$

14.2.1 Examples

1. Differentiate $y=(x^2+1)^{1/2}$.

Solution

Substitute for the inside function. Let $z=x^2+1$. Then $y=z^{\frac{1}{2}}$.

Use the chain rule above.

$$\frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}$$

$$= \frac{1}{2} z^{-\frac{1}{2}} \times 2x$$

$$\frac{dy}{dx} = x\left(x^2+1\right)^{-\frac{1}{2}}$$

2. Differentiate $x\left(x^2+1\right)^{1/2}$

Solution

This is a product . $u = x$ and $v = \left(x^2+1\right)^{1/2}$. Use the product rule of 14.1, with the derivative of v which was found in Example 1.

which was found in Example 1.

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= x \times x\left(x^2+1\right)^{-1/2} + \left(x^2+1\right)^{1/2} \times 1$$

$$\frac{dy}{dx} = x^2\left(x^2+1\right)^{-1/2} + \left(x^2+1\right)^{1/2}$$

14.2.2 Exercises

1. Differentiate the following:
   (a) $y = \left(3x+7\right)^{1/2}$
   (b) $y = \left(2-x^2\right)^5$
   (c) $y = \sqrt{\left(1+2x^2\right)}$
   (d) $y = \dfrac{1}{\sqrt{\left(1+x\right)}}$
   (e) $y = \sqrt{\left(1+x+x^2\right)}$
   (f) $y = \left(1+x^2\right)^{100}$
2. Differentiate the following :
   (a) $y = x\left(1+2x\right)^{1/2}$
   (b) $y = \left(x+3\right)\left(x-17\right)^{21}$
   (c) $y = \dfrac{x+4}{\sqrt{\left(x^2+1\right)}}$
   (d) $y = \dfrac{\sqrt{x}}{\sqrt{\left(x+1\right)}}$
3. Let $y = \sqrt{\left(x^2+3\right)}$. Find the equations of the tangent and normal to the curve at the point $\left(1,2\right)$.

14.3 Differentiation of trig and log functions

Provide that the angles are measured in radians, the trigonometric functions as differentiated as followings:

If $y = \sin x, \dfrac{dy}{dx} = \cos x$

If $y = \cos x, \dfrac{dy}{dx} = -\sin x$

If $y = \tan x, \dfrac{dy}{dx} = \sec^2 x$

e is a number approximately equal to 2.718281828. The exponential function is defined as e^x.

$\ln x$ is defined as $\log_e x$.

e^x and $\ln x$ can be differentiated as follows:

If $y = e^x, \dfrac{dy}{dx} = e^x$

If $y = \ln x, \dfrac{dy}{dx} = \dfrac{1}{x}$.

14.3.1 Examples

1. Differentiate $y = e^x \cos x$

Solution

This is a product, so the product rule must be used.

$$\frac{dy}{dx} = e^x \cos x + e^x(-\sin x)$$

$$\frac{dy}{dx} = e^x(\cos x - \sin x)$$

2. Differentiate $y = \ln(1 + x^2)$

Solution

This is a function of a function, so the chain rule is used.

$$\frac{dy}{dx} = \frac{1}{1+x^2} \times 2x$$

$$\frac{dy}{dx} = \frac{2x}{1+x^2}$$

14.3.2 Exercises

1. Differentiate the following:
   (a) $\sin 3x$
   (b) $\cos 4x$
   (c) e^{2x-1}
   (d) $\tan(5x-3)$

(e) $5\cos 3x - 2\sin 4x$

(f) $e^{\cos x}$

(g) $\ln(2x-3)$

(h) $\ln(\cos x)$

(i) $x\cos x$

(j) $x^2 \sin x$

(k) $e^x \tan x$

(l) $x\ln x$

(m) $e^x \ln x$

(n) $e^{-x}\sin 2x$

(o) $(\cos 3x)(\sin 2x)$

(p) $\cos(1+x^2)$

(q) $x\sin(1+3x^2)$

(r) $\cos^2 x$

(s) $\sin^3 3x$

(t) $\dfrac{x}{\cos x}$

(u) $\dfrac{x^2}{\cos x + \sin x}$

(v) $\dfrac{\cos^2 x}{\ln x}$

(w) $\cos\sqrt{(x^2+3)}$

(x) $e^x(\cos x - \sin x)$

(y) $\ln 3x - \ln x$

(z) $e^{\ln x}$

2. By considering $\tan x = \dfrac{\sin x}{\cos x}$, use the quotient rule to prove that $\dfrac{d}{dx}(\tan x) = \sec^2 x$

3. Find the derivatives of $\cot x, \sec x, \cos ecx$.

4. Use the fact that $10^x = e^{x\ln 10}$

5. Use the fact that $\log_{10} x = \dfrac{\ln x}{\ln 10}$ to differentiate $\log_{10} x$.

6. Let a be positive constant. Differentiate a^x and $\log_a x$

7. Find the maxima and minima of the following:

(a) $x^2 e^x$

(b) $\ln(1+x+x^2)$

14.4 Applications of the chain rule

An explicit function gives y directly in terms of x. An implicit function connects y and x together by an equation.

$y=3x^2+7x-3$ and $y=\cos 3x-7e^x$ are explicit .

$x^2+y^2=3$ and $\cos xy+7y^2=8x$ are implicit.

If x and y are both expressed, not in terms of each other, but in terms of a third variable t, then t is a parameter.

A small change in x is often written as δx. (Pronounced delta x). Small changes are connected by the following approximation:

$$\delta y \approx \frac{dy}{dx}\times\delta x$$

14.4.1 Examples

1. Find the gradient of the curve $y^2+3xy+2x^2=6$ at the point $(1,1)$.

   Solution

   Here y is an implicit function of x. The y^2 term and the $3xy$ term are differentiate as follows:

   $\frac{d\left(y^2\right)}{dx}=\frac{d\left(y^2\right)}{dy}\times\frac{dy}{dx}=2y\frac{dy}{dx}$ (Using the Chain Rule)

   $\frac{d(3xy)}{dx}=y\frac{d3x}{dx}+3x\frac{dy}{dx}=3y+3x\frac{dy}{dx}$ (Using the Product Rule)

   Go through the equation, differentiating each term.

   Substitute the values $x=1$ and $y=1$.

   $$2\frac{dy}{dx}+3+3\frac{dy}{dx}+4=0$$

   $$\frac{dy}{dx}=-\frac{7}{5}$$

2. Variables x and y are given by $x=t^2+1, y=t^3+2$. Find $\frac{dy}{dx}$ in terms of t.

   Solution

   Use the chain rule :

   $$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{dy}{dx}\div\frac{dx}{dt}=3t^2\div 2t$$

   $$\frac{dy}{dx}=\frac{3}{2}t$$

3. The radius of a sphere is 12 cm. What is the approximate change in surface area if the radius increases by $0.01cm^2$?

   Solution

The area is given by $A = 4\pi r^2$. Use the small change formula above, with $\delta r = 0.01$.

$$\delta A \approx \frac{dA}{dr} \times \delta r = 8\pi r \times 0.01 = 8\pi \times 12 \times 0.01$$

The change is approximately $3cm^2$

4. Air is pumped into a spherical balloon at $10cm^3s^{-1}$. How fast is the radius increasing when it is 8 cm?

Solution

The rate of change of the volume is $\frac{dV}{dt} = 10$.

The rate of change of the radius is $\frac{dr}{dt}$.

The volume is given by $V = \frac{4\pi r^3}{3}$.

Use the chain rule: $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi r^2 \times \frac{dr}{dt}$

$$\frac{dr}{dt} = \frac{10}{4\pi 8^2}$$

The radius is increasing at 0.012 cm per second

14.4.2 Exercises

1. Find the gradients of the following functions at the points shown.

   (a) $x^2 + y^2 = 5$ at $(1,2)$

   (b) $x^2 - y^3 = 1$ at $(3,2)$

   (c) $y^2 - 3x + 2y + x^2 = 1$ at $(2,1)$

   (d) $x\cos y = 1$ at $(1,0)$

   (e) $y^2 - 2xy = -3$ at $(2,3)$

   (f) $yx^2 + x^3 + 2y = 2$ at $(2,-1)$

   (g) $ye^x + e^y + e^x = 2$ at $(0,0)$

   (h) $y^2x^2 + 2x - 3y = 0$ at $(1,2)$

2. For each of the following function, express $\frac{dy}{dx}$ in terms of x and y.

   (a) $x^2 + y^2 = 4$

   (b) $x^2 + y^3 - 3x + y = 5$

   (c) $x^2 + 4xy - y^2 = 3$

   (d) $x^2 + 3yx^3 + 4y = 7$

3. Show that the stationary points of the ellipse $2x^2 + xy + y^2 = 1$ occur when $y + 4x = 0$.

4. Find the equation of the tangent to $5x^2 - 4y^2 = 1$ at the point $(1,1)$.
5. Find the maximum and minimum values of x for the curve $x^2 - xy + y^2 = 1$.
6. Find the maximum and minimum values of x for the curve in Question 5. (Hint : when x is a maximum and minimum , $\frac{dx}{dy} = 0$)
7. Find $\frac{dy}{dx}$ in terms of the parameter t for the following functions:
   (a) $x = t^2, y = 2t^3 + 1$
   (b) $x = \frac{1}{1+t}, y = t^2$
   (c) $x = t + 2t^2, y = t^2 - t^3$
   (d) $x = \frac{1}{1+t}, y = \frac{1}{1+t^2}$
   (e) $x = at^2, y = 2at$ (a is constant)
   (f) $x = ct, y = \frac{c}{t}$ (c is constant)
   (g) $x = \cos t, y = \sin t$
   (h) $x = 3\cos t, y = 2\sin t$
   (i) $x = e^t + e^{-t}, y = e^t - e^{-t}$
   (j) $x = 2\tan t, y = \frac{3}{\cos t}$
8. If x and y are given parametrically by $x = 3\cos t$ and , find the equation of the tangent at the point where find the equation of the tangent at the point where $t = \frac{1}{4}\pi$.
9. Find the equation of the normal to the curve given by $x = t^2 + 1, y = t^3 - 1$, at the point where $t = 2$.
10. Find the approximate increase in the area of a square when its side changes by 0.02 cm, when the side is 10 cm.
11. A cube has side 3 cm. Find the approximate change in radius if the volume changes by 0.02 cm, when the side is 10 cm.
12. The radius of a sphere is 5 cm. Find the approximate change in radius if the volume changes by $0.1cm^3$.
13. The radius if a sphere is measured as 12.3 cm, with a possible error of 0.05 cm. What is the possible error in the volume?
14. The surface area of a sphere is measured as $67\ cm^2$, with a possible error of approximate error in the volume?
15. The side of a square is measured, with a possible error of 1%. What was the approximate percentage error in the area?

16. The percentage error when measuring the are of a circle was 4%. What was the approximate percentage error in its radius?
17. The side of a sphere is increasing at $0.5\ cms^{-1}$. At what rate is the area increasing at a time when the side is 15 cm?
18. The radius of a sphere is increasing at $0.02\ cms^{-1}$. At what rate is the surface area increasing when the radius is $20\ cm^2s^{-1}$. At what rate is the volume increasing ?
19. The area of a circle is increasing at $20\ cm^2s^{-1}$. At what rate is the radius increasing, when it is 45 cm?
20. Water is poured into a cone, of semi-vertical angle $45°$, at $10\ cm^3s^{-1}$. When the height of the water is 15 cm, at what rate is it increasing ?
21. The volume of a cube is decreasing at $3\ cm^3s^{-1}$. When the side is 4 cm, what is the rate of decrease of (a) the side (b) the surface area?
22. At the equinox on the equator the angle θ of elevation of the sun changes at 0.000073 radians/sec. A flagpole of height 40 m. throws a shadow of length $40/\tan\theta$. Find the rate of decrease of the shadow when $\theta = 0.1$.

14.5 Examination questions

1. (a) Differentiate with respect to x,

   (i) $\ln(3x+1)$ (ii) $x\cos 2x$

   (b) Find the equation of the tangent to the curve

   $$y = \frac{3+x}{1-2x}$$

   at the point where the curve crosses the line $y = -1$.

   (c) Given that $x^2y + y^2 = 10$, find $\frac{dy}{dx}$ in terms of x and y.

2. (a) sketch the curve $y = \log_e(x+3)$.

   (b) Find the gradient of this curve at point where $x = e-3$. Hence show that the equation of the tangent to the curve at this point is $ey = x+3$.

3. The surface area of an expanding spherical balloon is increasing at the rate of 16 cm^2/s. At the instant when the surface area is $144\ cm^2$, calculate

   (a) The rate , in cm/s to 2 significant figures, at which the radius of the balloon is increasing,

   (b) The rate , in cm^3/s to significant figures, at which the volume of balloon is increasing.

   Find , in cm^3 to 2 significant figures , the approximate increase in the volume of the balloon when the surface area increase from $144\ cm^3$ to $146\ cm^2$.

4. Express $\frac{3x+16}{(x-3)(3x+2)}$ in partial fractions.

   Hence find the value of $\frac{d}{dx}\left[\frac{13x+6}{(x-3)(3x+2)}\right]$ when $x = 2$.

5. (a) A curve has the equation $x^2 - xy + y^3 = k$,where k is a constant.

Find $\frac{dy}{dx}$ in terms of x and y.

Prove that the curve cannot posses a tangent which is parallel to the y-axis if $k < -\left(\frac{\pi}{2} - \theta\right)\frac{1}{27}$.

(b)If $y = \frac{e^x}{1+x^2}$,prove that the $\frac{dy}{dx}$ may be written in the form $\frac{(x-1)^2 e^x}{\left(1+x^2\right)^2}$

Hence , or otherwise , prove that ,for all positive values of x,

$\frac{e^x}{1+x^2} > 1$

6. (a) the parametric equations of a curve are : $x = a\left(1+\frac{1}{t}\right), y = a\left(t - \frac{1}{t^2}\right)$, where a is a constant and $t \neq 0$.

Find the equation of the normal to the curve at the point where $t = 2$.

(b)the equation of a curve is $(y-x)^2 = 2a(y+x)$, where a is a constant .

Find an expression for $\frac{dy}{dx}$ in terms of x, y and a.

(c)the velocity v of a point moving in a straight line is given in terms of the time t by $v = e^{-2t}\sin 3t$.

Find the smallest positive value of t for which the acceleration of the point is zero, giving two significant figure in your answer.

7. Apply the small increment formula $f(x+\delta x) - f(x) \approx \delta x f'(x)$ to $\tan x$ to find are approximate value of

$\tan\left(\frac{100\pi + 4}{400}\right) - \tan\frac{\pi}{4}$

8. Sketch the curve whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a, b > 0)$

Find the gradient of the curve at the point $P\left(a\cos\theta, b\sin\theta\right)$.

Hence show that the equation of the tangent at P is $bx\cos\theta + ay\sin\theta = ab$ and that the equation of the normal at P is

$ax\sin\theta - by\cos\theta = \left(a^2 + b^2\right)\sin\theta\cos\theta$.

The tangent at P meets the x-axis at T and the y-axis at U ; the normal at P meets the x-axis at M and the y-axis at N . Show that NT is perpendicular to MU .

Write down the coordinates of the mid-point Q of UT and hence find the equation of the locus of Q as θ varies.

Common errors

1. **Products and quotients**

   The derivative of a product is not the product of the derivatives. Similarly the derivative of a quotient is not the quotients of the derivatives. In both case the rules must be used.

   If $y = uv$, then $\frac{dy}{dx} \doteq \frac{du}{dx} \times \frac{dv}{dx}$

   If $y = \frac{u}{v}$, then $= \frac{dy}{dx} \neq \frac{du}{dx} \div \frac{dv}{dx}$

2. **Chain rule**

   When using the chain rule

   $$\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx}$$

   Make sure you put the value of z in the $\frac{dy}{dx}$ term. Watch out for the following :

   For $y = \sin x^2$, $\frac{dy}{dx} \neq 2x \cos x$. It should be $2x \cos x^2$.

   Once you have found the derivative, substitute back so that the answer is in terms of x. It would be incorrect to leave z in the answer.

3. **Differentiating power functions**

   The function a^x is very different from x^a. Do not differentiate as though they were the same function. In particular:

   If $y = e^x$, then $\frac{dy}{dx} \neq xe^{x-1}$. It should be e^x.

4. **Implicit functions**

   Many difficulties are found when differentiating implicit functions.

   For something like y^2, use $\frac{d\left(y^2\right)}{dx} = \frac{d\left(y^2\right)}{dx} \times \frac{dy}{dx}$.

   For something like .

   For something like $3xy$, use the product rule. And if there is a constant at the right hand side , its derivative is 0. Note the following :

   If $x^2 + y^2 = 3$, then $2x + 2y \times \frac{dy}{dx} \neq \frac{dy}{dx}$.

   The right hand side of the last equation should be 0.

Solution (to exercise)

14.1.2

1. (a) $\left(x^2 + 7x - 3\right) + x\left(2x + 7\right)$ (b) $\left(2x^3 + 6x\right) + \left(x + 3\right)\left(6x^2 + 6\right)$

   (c) $2x\left(3x^5 - 3x^2 + x\right) + x^2\left(15x^4 - 6x + 1\right)$ (d) $2x\left(x^2 + 1\right) + 2x\left(x^2 - 1\right)$

2. (a) $\frac{(x+3)-x}{(x+3)^2}$ (b) $\frac{(2x+7)(x^2+3)-2x(x^2+7x-2)}{(x^2+3)^2}$

(c) $\frac{6x(x-17)-3x^2}{(x-17)^2}$ (d) $\frac{-1}{(x+3)^2}$

3. $y=4.3y+4x=27$

4. (a) $(0,0)$ max, $(2,4)$ min (b) $(0.414, 0.828)$ min, $(-2.414, -4.828)$ max

(c) $(0,1)$ max (d) $\left(-2, \frac{1}{5}\right)$ max

14.2.2

1. (a) $1\frac{1}{2}(3x+7)^{-1/2}$ (b) $-10x(2-x^2)^4$

(c) $2x(1+2x^2)^{-1/2}$ (d) $-\frac{1}{2}(1+x)^{-3/2}$

(e) $\frac{1}{2}(2x+1)(1+x+x^2)^{-1/2}$ (f) $200x(1+x^2)^{99}$

2. (a) $(1+2x)^{1/2}+x(1+2x)^{-1/2}$ (b) $(x-17)^{21}+21(x+3)(x-17)^{20}$

(c) $\frac{\sqrt{(x^2+1)}-x(x+4)(x^2+1)^{-1/2}}{x^2+1}$ (d) $\frac{x^{-1/2}(\sqrt{x+1})-1}{2(\sqrt{x+1})^2}$

3. $y=\frac{1}{2}x+1\frac{1}{2}.y+2x=4$

14.3.2

1. (a) $3\cos x3x$ (b) $-4\sin 4x$ (c) $2e^{2x-1}$ (d) $5\sec^2(5x-3)$

(e) $-15\cos 3x-8\cos 4x$ (f) $-\sin xe^{\cos x}$ (g) $\frac{2}{2x-3}$

(h) $-\tan x$ (i) $\cos x - x\sin x$ (j) $2x\sin x + x^2\cos x$ (k) $e^x(\tan x+\sec^2 x)$

(l) $\ln x+1$ (m) $e^x \ln x+\frac{e^x}{x}$ (n) $-e^{-x}\sin 2x+2e^{-x}\cos 2x$

(o) $-3\sin 3x\sin 2x+2\cos 3x\cos 2x$ (p) $-2x\sin(1+x^2)$

(q) $\sin(1+3x^2)+6x^2\cos(1+3x^2)$ (r) $-2\cos x\sin x$

(s) $9\cos 3x\sin^2 3x$ (t) $\frac{\cos x+\sin x}{\cos^2 x}$

(u) $\frac{2x(\cos x+\sin x)-x^2(\cos x-\sin x)}{(\cos x+\sin x)^2}$ (v) $\frac{-2\cos x\sin x\ln x-\frac{\cos^2 x}{x}}{(\ln x)^2}$

(w) $-x(x^2+3)^{-1/2}\sin\sqrt{(x^2+x)}$ (x) $-2e^x\sin x$

(y) 0 (z) 1

3. $-\cos ecx, \sec x \tan x., -\cot x \cos ecx$

4. $(\ln 10)10^x$

5. $\frac{1}{x\ln 10}$

6. $(\ln a)a^x, \frac{1}{x\ln a}$

7. (a) $(0,0)$ min . $(-2,4)e^{2x}$ max. (b) $-\frac{1}{2}, \ln\frac{3}{4}$ min.

14.4.2

1. (a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $-\frac{1}{4}$ (d) ∞

   (e) 3 (f) $-1\frac{1}{3}$ (g) $-\frac{1}{2}$ (h) -10

2. (a) $-\frac{x}{y}$ (b) $\frac{3-2x}{3y^2+1}$ (c) $\frac{2x+4y}{2y-4x}$

4. $y = 1\frac{1}{4}x - \frac{1}{4}$

5. $\pm\frac{2}{\sqrt{3}}$

6. $\pm\frac{2}{\sqrt{3}}$

7. (a) $3t$ (b) $-2t1+t)^2$ (c) $\frac{2t-3t^2}{1-4t}$ (d) $\frac{-2t1+t)^2}{(1+t^2)^2}$

   (e) $\frac{1}{t}$ (f) $-\frac{1}{t^2}$ $-\cot$ (g) $-\cot t$ (h) $-\frac{2}{3}\cot t$

   (i) $\frac{e^t - e^{-t}}{e^t - e^{-t}}$ (j) $1\frac{1}{2}\sin t$

8. $3x + 4y = 12\sqrt{2}$

9. $3y + x = 26$

10. $0.4cm^2$

11. $0.27cm^3$

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References:

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More: (in Thai)

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