EP-Program - Strisuksa School - Roi-et

15 Math : Integration

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15 Integration

Integration is the opposite operation to differentiation.

If $\frac{d(F(x))}{dx} = f(x)$, then $\int f(x)dx = F(x) + c$.

The constant c is called the constant of integration.

15.1 Integration of powers of x

Powers of x are integrated by the rule:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

For all except $n = -1$

If functions are added or multiplied by constants, then the integrals are added or multiplied by constants.

$$\int (ax^n + bx^m)dx = \frac{ax^{n+1}}{n+1} + \frac{bx^{m+1}}{m+1} + c.$$

In particular, $\int 0dx = c; \int kdx = kx + c$, where k is a constant.

15.1.1 Examples

1. If $\frac{dy}{dx} = 6x^2 + 2$, find y given that $y = 7$ when $x = 1$.

Solution

Integrate $6x^2 + 2$.

$y = \int (6x^2 + 2)dx = 2x^3 + 2x + c.$

Substitute x=1 and y=7 to find that c=3.

$y = 2x^3 + 2x + 3.$

2. Integrate $(\sqrt{x} + 1)(x^2 - 3)$.

Solution

The bracket must be multiplied out before integration.

$$(\sqrt{x} + 1)(x^2 - 3) = x^{\frac{5}{2}} - 3x^{\frac{1}{2}} + x^2 - 3.$$

$$\int (\sqrt{x} + 1)(x^2 - 3)dx = \frac{x^{\frac{7}{2}}}{\frac{7}{2}} - \frac{3x^{\frac{7}{2}}}{\frac{3}{2}} + \frac{x^3}{3} - 3x + c$$

$$=\frac{2x^{\frac{7}{2}}}{7}-2x^{\frac{3}{2}}+\frac{x^3}{3}-3x+c$$

15.1.2 Exercises

1.For each of the following find y as a function of x.

(a) $\frac{dy}{dx}=2x,\ \ y=1 \text{ for } x=0$

(b) $\frac{dy}{dx}=3x^2+x,\ \ y=1 \text{ for } x=1$

(c) $\frac{dy}{dx}=x^2+x+1,\ \ y=1 \text{ for } x=1$

(d) $\frac{dy}{dx}=\frac{1}{x^2},\ \ y=1 \text{ for } x=1$

(e) $\frac{dy}{dx}=x^{3/2},\ \ y=1 \text{ for } x=4$

(f) $\frac{dy}{dx}=\frac{3}{\sqrt{x}},\ \ y=4 \text{ for } x=16$

(g) $\frac{dy}{dx}=x(x^2+1),\ \ y=0 \text{ for } x=1$

(h) $\frac{dy}{dx}=\frac{x^2+3}{\sqrt{x}},\ \ y=3 \text{ for } x=4$

2.Evaluate the following integrals:

(a) $\int 2x dx$

(b) $\int (3x^2+2x+1)dx$

(c) $\int (x^2+2x^3+5)dx$

(d) $\int (2x^5-\frac{5}{2}x^7)dx$

(e) $\int x^{3/2}dx$

(f) $\int (3-\sqrt{x})dx$

(g) $\int \frac{1}{x^3}dx$

(h) $\int (\frac{2}{x^2}+\frac{7}{x^3}+2)dx$

(i) $\int x(3x^2+7)dx$

(j) $\int (x-3)(x^2+2x)dx$

(k) $\int x(4x^2-2x^3)dx$

(l) $\int (\sqrt{x}+\frac{1}{\sqrt{x}})(x^4-3)dx$

15.2 Definite integrals and areas

When $f(x)$ is integrated to find $F(x)$, the result is the indefinite integral of f.

The difference of $F(x)$ between two values of x is the definite integral of f

If $\int f(x)dx=F(x)$, then:

$$\int_a^b f(x)dx=\left[F(x)\right]_a^b=F(b)-F(a)$$

The area bounded by a curve $y=f(x)$, the x-axis and the limits $x=a$ and $x=b$ is given by the definite integral of $f(x)$ between a and b.

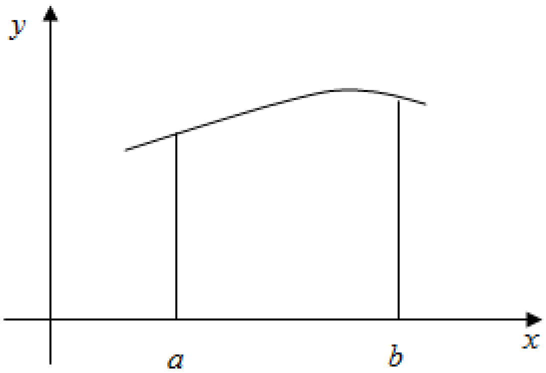


Fig 15.1

$$Area = \int_a^b f(x)dx$$

Negative area corresponds to a region below the x-axis.
If x and y are expressed in term of a parameter, then integrals can be found from the following:

$$\int ydx = \int y\frac{dx}{dt}dt.$$

15.2.1 Examples

1. Evaluate the definite integral $\int_1^2 (x^2 - \frac{3}{x^3})dx$

Solution
First find the indefinite integral.

$$\int (x^2 - \frac{3}{x^3})dx = \frac{1}{3}x^3 + \frac{3}{2}x^{-2} + c.$$

Evaluate this at the values 1 and 2 and subtract.

$$\int_1^2 (x^2 - \frac{3}{x^3})dx = (\frac{1}{3}\times 2^3 + \frac{3}{2}\times 2^{-2} + c) - (\frac{1}{3}\times 1^3 + \frac{3}{2}\times 1^{-2} + c)$$

$$\int_1^2 (x^2 - \frac{3}{x^3})dx = 1.2083$$

2.Find the area enclosed by the curve $y = \sqrt{x}$, the x-axis and the line $y = x - 2$.

Solution
A rough sketch of the area is shown. The curve and the line cross at (4,2).The area is the difference between the area under the curve and the area under the line.

$$Area = \int_0^4 \sqrt{x}dx - \int_2^4 (x-2)dx = \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^4 - \left[\frac{1}{2}x^2 - 2x\right]_2^4$$

$$Area = 3\frac{1}{3}$$

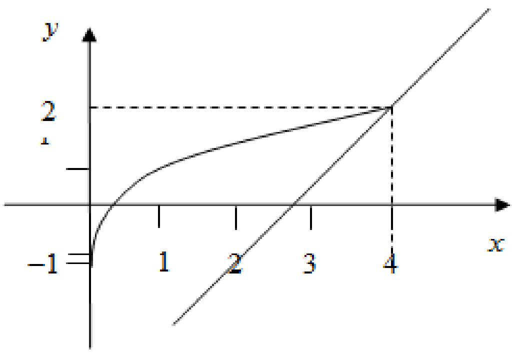


Fig 15.2

15.2.2 Exercises

1. Evaluate the following definite integrals:

(a) $\int_0^3 (2x+3x^2)dx$ (b) $\int_0^1 (x^2-\frac{1}{4}x^4)dx$

(c) $\int_1^9 (\sqrt{x}-\frac{3}{x^2})dx$ (d) $\int_1^4 x(x+3)dx$

(e) $\int_1^4 \frac{x^2-1}{\sqrt{x}}dx$ (f) $\int_{-1}^2 (x^3+2)^2dx$

2 .Find the areas enclosed by the following curves and the x-axis, between the limits shown:

(a) $y=x^2, x=1$ and $x=3$ (b) $y=\frac{1}{x^2}$, $x=1$ and $x=20$

(c) $y=x^{1/2}$, $x=0$ and $x=9$ (d) $y=x^3-x$, $x=0$ and $x=1$

3. Sketch the curve $y=3-2x-x^2$. Find the area of the region enclosed by the curve and the x-axis.

4. Sketch the curve $y=x^3-x.$ Find the areas of the region:

(a) enclosed by the curve above the x-axis
(b) enclosed by the curve below the x-axis. What do you notice?

5. On the same graph sketch the curve $y=1-x^2$ and the line $y=1-x$. Find the areas

(a) between the curve and the line
(b) enclosed between the curve, the line and the x-axis.

6. Find the area between the y-axis, the curve $y=x^3-4$ and the line $y=4$.

7. Find the areas of the two regions between the x-axis and the curve $y=(x+1)(x-3)(2x+1)$

8. For each of the following, evaluate the definite integral and sketch the area to which it corresponds.

(a) $\int_0^4 3x^2dx$ (b) $\int_1^2 \frac{4}{x^3}dx$

(c) $\int_0^2 (4-x^2)dx$ (d) $\int_1^9 x^{1/2}dx$

9. The variables x and y are given in terms of the parameter t by $x=t^2+1, y=t^3-t$. Find the area enclosed by the curve between $t=0$ and $t=1$ and the x-axis.

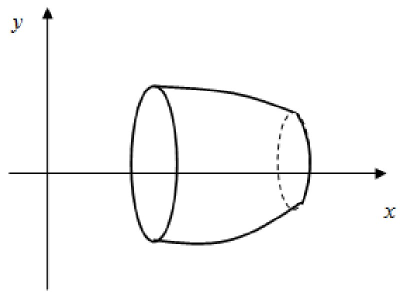
10. The variables x and y are given in terms of t by $x=t+\frac{1}{t}, y=t^2+\frac{1}{t^2}$. Find the area enclosed by the curve between $t=1$ and $t=2$ and the x-axis.

15.3 Volumes of revolution

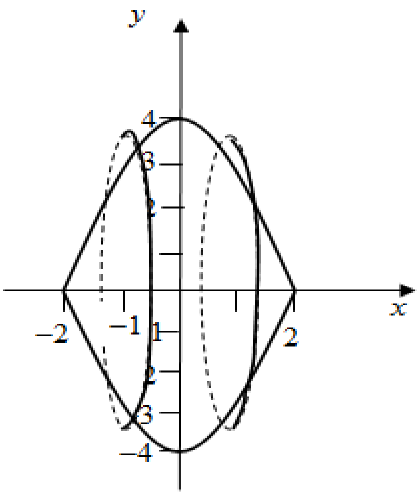
When a curve $y=f(x)$ is rotated through 360° about the x-axis, it passes through a region of space called a solid of revolution. The volume of this solid is the volume of revolution. The volume is given by:

$$V=\int \pi y^2 dx$$

Fig 15.3



15.3.1 Example



The area enclosed by the curve $y = 4 - x^2$ and the x-axis is rotated about the x-axis. Find the volume of revolution. (Fig 15.4)

Solution

The curve crosses the x-axis at $x = -2$ and $x = 2$. The volume is therefore:

$$V = \int_{-2}^{2} \pi (4 - x^2)^2 dx = \int_{-2}^{2} \pi (16 - 8x^2 + x^4) dx$$

$$V = \pi \left[16x - \frac{8x^3}{3} - \frac{x^5}{5} \right]_{-2}^{2}$$

$$V = 107.2$$

Fig 15.4

15.3.2 Exercises

1.Find the volumes enclosed when the following curves between the limits shown are rotated about the x-axis.

(a) $y = 3x^2$, x = 1 and x = 2 (b) $y = \sqrt{x}$,x = 0 and x = 3

(c) $y = \frac{1}{x}$, x = 1 and x = 10 (d) $y = 2x - 3$, x = 2 and x = 4

(e) $y = 3x - \frac{2}{x}$, x = 1 and x = 4 (f) $y = \sqrt{(1 + x^2)}$, x = 0 and x = 3

2. The line $y = 3x$, between $x = 0$ and $x = 4$, is rotate about the x-axis. Show that the solid of revolution is a corn. What is its height and its base radius? Find the volume of revolution, and verify that this agrees with the formula $V = \frac{1}{3}\pi r^2 h.$

3. By rotating the line $y = rx/h$, from $x = 0$ to $x = h$, about the x-axis, prove that the volume of a cone of height h and base radius r is $\frac{1}{3}\pi r^2 h.$

4. The circle with centre at the origin and radius 1 has equation $y = \sqrt{(1 - x^2)}$.what is the solid formed when this curve is rotated about the x-axis? Find the volume of revolution, and verify that it agrees with the formula for the volume of this solid.

5. Find the equation of the straight line through $(0,1)$ and $(2,3)$. When this line is rotated about the x-axis, the solid of revolution is a frustum, of height 2 and radii 1 and 3.Find the volume of the frustum.

6. A curve is given parametrically by $x = at^2, y = 2at$. Find the area enclosed by the curve and the line $x = a$. Find the volume when this area is rotated about the x-axis.

15.4 Examination questions

1. (a) Integrate with respect to x:

   (i) $x^2 - x^{3/2}$ (ii) $(x^2-1)^2$

   (b) A curve passes through the point $(9,2)$ and its gradient at the point (x,y) is given by $\frac{dy}{dx} = \sqrt{x}+3$. Find the equation of the curve and the equation of the tangent at the point $(9,2)$.

2. The finite region contained between the x-axis and the curve whose equation is $y = x(2-x)$ is completely rotated about the x-axis. Calculate the volume of the solid so formed. (Leave your answer as a multiple of π)

3. (a) Using a scale of 2 cm to 1 unit on each axis, sketch the graph of $y = 3 - \frac{5}{x^2}$, for $1 \le x \le 5$.

   (b)Shade the area represented by I when $I = \int_2^4 (3 - \frac{5}{x^2})dx$

   (c) The rate of flow of water through a drainpipe t minutes after the beginning of a storm is approximated by $100(3 - \frac{5}{t^2})m/\min$ for $t \ge 2$. If the drainpipe remains full and has cross-sectional area $0.2m^2$, calculate the volume of water flowing through the drain during the time from $t = 2$ to $t = 2$.

4. Sketch the curve $y = 3x^{2/3}$ for values of x from 1 to 8. A vase is made from thin glass in the shape of the surface obtained by rotating this curve, together with the line segment from $(0,3)$ to $(1,3)$ about the y-axis which is vertical. The unit along each axis is 1 cm. Water is poured into the vase at a constant rate of 30 cm^3s^{-1}. Prove by integration that the volume, V cm^3, of water in the vase, when its depth is h cm, is given by

$$V = \frac{\pi}{108}\left[(h+3)^4 - 3^4\right].$$

(a)How long does it take for the depth to increase from 0 to 6 cm?

(b) Find, when $h = 6$, the rate at which

   (i) the water level is rising,

   (ii) the area of the horizontal water surface is increasing.

5. A curve is defined by the equations $x = t^2 - 1, y = t^3 - t$, where t is a parameter.
   Sketch the curve for all real values of t.
   Find the area of the region enclosed by the loop of the curve.

6. Sketch the graph of the function $f : x \to |3x-2|$ in the interval $0 \le x \le 1$.

Evaluate $\int_0^1 f(x)dx$ [If you wish, you may use a formula for the area of a triangle.]

1. Integration

(a) Do not integrate when you are asked to differentiate, or the other way round.

(b) When integrating a negative power of x, you still add 1 to the power.

$$\int x^{-3}dx = -\frac{1}{2}x^{-2} + c, not -\frac{1}{4}x^{-4} + c.$$

(c) Do not forget to divide by $(n+1)$ when integrating x^n. And divide by $(n+1)$, not by n.

(d) If functions are multiplied, you cannot integrate them individually and multiply the results.

$$\int f(x)g(x)dx \neq \int f(x)dx \times \int g(x)dx.$$

A similar mistake arises when one function is divided by another.

(e) Do not forget the constant of integration c.

2. Definite integrals.

Make sure that you get the limits of the integration the correct way round. Be very careful of negative signs when subtracting the second value from the first.

3. Areas

Question often ask you to find the area between two curves. You must first find out where they intersect.

4. Volume

(a) When using the formula $\int \pi y^2 dx$, you must square first and then integrate. Watch our for the following:

$\int \pi y^2 dx \neq \pi(\int ydx)^2$

(b) When using the formula $\int \pi y^2 dx$, you must express y as a function of x. And the other way round: if a curve is rotated about the y-axis, you will use the formula $\int \pi x^2 dy$.

Write x as a function of y before integrating.

Solution (to exercise)15

15.1.2

1. (a) $y = x^2 + 1$ (b) $y = x^3 + \frac{1}{2}x^2 - \frac{1}{2}$ (c) $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 + x + \frac{7}{6}$

(d) $2 - \frac{1}{x}$ (e) $y = \frac{2}{5}x^{5/2} - 11.8$ (f) $y = 6\sqrt{x} - 20$

(g) $\frac{1}{4}x^4 + \frac{1}{2}x^2 - \frac{3}{4}$ (h) $y = \frac{2}{5}x^{5/2} + 6\sqrt{x} - 21.8$

2. (a) $y = x^2 + c$ (b) $x^3 + x^2 + x + c$ (c) $\frac{1}{3}x^3 + \frac{1}{2}x^4 + 5x + c$

(d) $\frac{1}{3}x^6 - \frac{5}{16}x^8 + c$ (e) $\frac{2}{5}x^{5/2} + c$ (f) $3x - \frac{2}{3}x^{3/2} + c$

(g) $c - \frac{1}{2}x^2$ (h) $-\frac{2}{x} - \frac{7}{2x^2} + 2x + c$ (i) $\frac{3}{4}x^4 + \frac{7}{2}x^2 + c$

(j) $\frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2 + c$ (k) $x^4 - \frac{2}{5}x^5 + c$

(l) $\frac{x^{5.5}}{5.5} + \frac{x^{4.5}}{4.5} - 2x^{1.5} - 6x^{0.5} + c$

15.2.2

1. (a) 36 (b) 0.283 (c) $14\frac{2}{3}$ (d) $43\frac{1}{2}$

(e) 10.4 (f) 45.43

2. (a) $8\frac{2}{3}$ (b) 0.95 (c) 6 (d) $-\frac{1}{4}$
3. $10\frac{2}{3}$
4. (a) $\frac{1}{4}$ (b) $-\frac{1}{4}$
5. (a) $\frac{1}{6}$ (b) $\frac{7}{6}$
6. 12
7. $0.15625, -32.15625$
8. (a) 64 (b) 1.5 (c) $5\frac{1}{3}$ (d) 4
9. $\frac{4}{15}$
10. $\frac{37}{24}$

15.3.2

1. (a) 175.3 (b) 14.1 (c) 2.83 (d) 64.9
   (e) 490 (f) 37.7
2. $4, 12, 603$
4. Sphere. 4.189
5. $y = x + 1, 27.2$
6. $2, \frac{2}{3}a^2, 2\pi a^3$

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References:

Solomon, R.C. (1997), *A Level: Mathematics* (4th Edition) , Great Britain, Hillman Printers(Frome) Ltd.

More: (in Thai)
http://home.kku.ac.th/wattou/service/m456/09.pdf