16 Techniques of Integration

Trigonometric and exponential functions are integrated as follows:

$\int \cos x dx = \sin x + c$ $\int \sin x dx = -\cos x + c$

$\int e^x dx = e^x + c$ $\int \frac{1}{x} dx = \ln x + c$

16.1 Integration by substitution

Simple function can often be integration by inspection . More complicated functions which cannot be directly integration can be reduced to a simple form by *substitution* .

16.1.1 Examples

1. Find the integral $\int \cos 3x dx$.

Solution

The answer must be of the form $k \sin 3x + c$, where k is a constant . The derivative of $\sin 3x$ is $3\cos 3x$. This is 3 times too great. Hence to balance out k must be $\frac{1}{3}$

$$\int cox3xdx = \frac{1}{3}\sin 3x + c$$

2. Evaluate the integral $\int_0^{\frac{\pi}{4}} \sin^3 x \cos x dx$.

Solution

$\sin^3 x$ is of the form $f(g(x))$, where $g(x) = \sin x$. Put $u = \sin x$. $\frac{du}{dx} = \cos x$,

$$\int u^3 du = \left[\frac{1}{4}u^4\right]_{x=0}^{x=\frac{\pi}{4}} = \left[\frac{1}{4}\sin^4 x\right]_{x=0}^{x=\frac{\pi}{4}}$$

The integral is $\frac{1}{4}\sin^4 \frac{\pi}{4} = \frac{1}{6}$

3. Find the indefinite integral $\int x\sqrt{(1+x)}dx$

Solution

Put $u = 1 + x$. Then $x = u - 1$. Also $\frac{du}{dx} = 1$, hence $du = dx$.

The integral becomes:

$$\int (u-1)\sqrt{u}\,du = \int \left(u^{3/2} - u^{1/2}\right) du = \frac{u^{3/2}}{5/2} - \frac{u^{3/2}}{3/2} + c$$

The integral is $\frac{2}{5}(1+x)^{5/2} - \frac{2}{3}(1+x)^{3/2} + c$

16.1.2 Exercise

1. Find the follow integrals by inspection:

   (a) $\int \sin 2x\,dx$ (b) $\int \cos\frac{1}{2}x\,dx$

   (c) $\int \sin(3x-1)\,dx$ (d) $\int 4\cos(2x+3)\,dx$

   (e) $\int e^{3x}\,dx$ (f) $\int e^{2x+1}\,dx$

   (g) $\int \frac{1}{x+3}\,dx$ (h) $\int \frac{1}{2x-5}\,dx$

Evaluate the following integrals:

2. $\int_0^{\frac{\pi}{2}} \cos^4 x \sin x\,dx$
3. $\int_0^4 x\sqrt{1+x^2}\,dx$
4. $\int_0^1 e^x\sqrt{2+e^x}\,dx$
5. $\int_1^2 xe^{x^2}\,dx$
6. $\int_0^{\frac{\pi}{6}} \cos xe^x\,dx$
7. $\int_0^a \frac{x}{a+x^2}\,dx$
8. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x}\,dx$
9. $\int_1^2 \frac{2x+1}{x^2+x+1}\,dx$
10. $\int x^2\sqrt{(x^3+4)}\,dx$
11. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$
12. $\int \sec^2 \tan^3 dx$
13. $\int \frac{1}{x\ln x}\,dx$
14. $\int x\sqrt{(3+x)}\,dx$
15. $\int \frac{2x}{\sqrt{(1+3x)}}\,dx$
16. $\int (x-1)(x+1)^{3/2}\,dx$
17. $\int \frac{1}{(3x-1)^3}dx$
18. $\int x(1+4x)^{14}dx$
19. $\int (x-3)(x+2)^{12}dx$
20. Use the identity $\cos^2 x = 1 - \sin^2 x$ to evaluate $\int \cos^3 x\,dx$
21. $\int \sin^5 x\,dx$
22. $\int \frac{dx}{1-e^x}$ (Multiply top and bottom by e^{-x})

23. $\int \cos^2 x dx$ (Use the identity $\cos 2x \equiv 2\cos^2 x - 1$)

24. $\int \sin^2 3x dx$ (Use the identity $\cos 2x \equiv 1 - 2\sin^2 x$)

25. Express $f(x) = \dfrac{1}{(1+x)(2+x)}$ in partial fractions. Hence find $\int f(x)dx$.

26. By using partial fractions, find the followings:

(a) $\dfrac{1}{(3+x)(2+x)}dx$ (b) $\dfrac{x+4}{(2+x)(1+x)^2}dx$

(c) $\dfrac{2-x}{(1+2x)(1-3x)}dx$

16.2 Integration by parts

A product of two functions can sometimes be integrated by means of *integration by parts.* Suppose that w is the derivative of u, i.e. that $w = \dfrac{du}{dx}$

Then:

$$\int wvdx = uv - \int u\frac{dv}{dx}dx$$

16.2.1 Examples

1. Find $\int xe^x dx$

Solution

Hence let v be x and w *be* e^x. Then $u = e^x$. Apply the formula:

$\int xe^x dx = xe^x - \int e^x - \int e^x dx$

$\int xe^x dx = xe^x - e^x + c$

2. Find $\int \ln x dx$

Solution

Regard $\ln x$ as $1 1\times \ln x$. Take w as 1 and v as $\ln x$. Then $u = x$. Apply the formula:

$\int 1\times \ln x dx = x\ln x - \int x\dfrac{d\ln x}{dx}dx = x\ln x - \int \dfrac{x}{x}dx$

$\int \ln x dx = x\ln x - x + c$

16.2.2 Exercises

Evaluate the following:

1. $\int xe^{-x}dx$ 2. $\int xcoxdx$

3. $\int x\sin xdx$ 4. $\int x\ln xdx$

5. $\int x^2 \ln xdx$ 6. $\int x^n \ln xdx$

7. $\int \sin^{-1} x dx$ (Use the fact that $\frac{d\left(\sin^{-1} x\right)}{dx} = \frac{1}{\sqrt{\left(1-x^2\right)}}$)

8. $\int \tan^{-1} x dx$ (Use the fact that $\frac{d\left(\tan^{-1} x\right)}{dx} = \frac{1}{1+x^2}$)

9. $\int x^2 e^x dx$

10. $\int x^2 e^{ax} dx$

11. $\int x^2 \cos 3x dx$

12. $\int e^x \left(x-3\right) dx$

13. $\int x^2 \sin ax dx$

14. $\int \frac{\ln x}{x^2} dx$

16.3 Examination questions

1. (a) Integrate with respect to x

(i) $\sin 4x$ (ii) $\frac{1}{2} cos\left(\frac{1}{2}x + \frac{1}{2}\right)$

(b) Evaluate $\int_0^1 \left(1+9x\right)^{-1} dx$

2. Draw a sketch to show the finite region bounded by the curve $y = e^{x/2}$ and the lines $x = 2, x = 4$ and $y = 0$.

Leaving your answer in terms of e, calculate the area of this region.

3. (a) (i) find

$$\int \frac{1}{x\left(x+1\right)} dx, \; x > 0$$

Using the substitution $u = e^x$ and the answer to (i) , or otherwise

(ii) find

$$\int \frac{1}{1+e^x} dx$$

(b) Use integration by parts to find

$$\int x^2 \sin x dx$$

4. (i) Find $\int \left(3x+4\right) e^{2x} dx$

(ii) By using the substitution $x = 2\tan\theta,$ evaluate $\int_0^2 \frac{2}{\left(4+x^2\right)^2} dx$

5. Using the substitution $x = 3 + \sin\theta,$ show that:

$$\int_2^4 \sqrt{\left(4-x\right)\left(x-2\right)} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\theta d\theta$$

Hence evaluate the integral

6. Sketch the graph of $y = \sin^{-1}\frac{x}{a},$ where $a > 0,$ stating the domain and the range of the function.

Using the fact that the derivative of $\sin x$ is $\cos x,$ show that

$$\frac{dy}{dx} = +\frac{1}{\sqrt{\left(a^2 - x^2\right)}}$$

Hence. Or otherwise, show that $\int_0^1 \frac{dx}{\sqrt{\left(4-\left(x+1\right)^2\right)}} = \frac{\pi}{3}$

Common errors

1. Substitution
   (a) When substituting u for x, do not forget to change the dx to a du by means of $\frac{du}{dx}$. You cannot integrate a function of u with respect to x.
   (b) When finding and indefinite integral by substituting u for x, you must re-substitute back at the end to get the answer in terms of x. Do not give the answer in terms of u
   (c) When finding a definite integral by substitution, remember that the limits refer to values of x . Do not apply these values to u when working out the final answer
2. Integration by parts
   When integrating something like xe^x , integrate the e^x term first. If you integrate the x term first, you will obtain $\frac{1}{2}x^2e^x$, Which is harder to integrate than the function you started with.

Solution (to exercise)

1. (a) $-\frac{1}{2}\cos 2x + c$ (b) $2\sin\frac{1}{2}x + c$ (d) $2\sin\left(2x+3\right) + c$
   (e) $\frac{1}{3}e^{3x} + c$ (f) $\frac{1}{2}e^{2x+1} + c$ (g) $\ln\left(x+3\right) + c$
   (h) $\frac{1}{2}\ln\left(2x+5\right) + c$
2. $\frac{1}{5}$
3. 23
4. 3.37
5. 25.9
6. 0.649
7. $\frac{1}{2}\ln\left(1+a\right)$
8. 0.693
9. 0.847
10. $\frac{2}{9}\left(x^3+4\right)^{3/2} + c$
11. $2\sin\sqrt{x} + c$
12. $\frac{1}{4}\tan^4 x + c$
13. $\ln\left(\ln x\right) + c$

14. $0.4(x+3)^{5/2}-2(x+3)^{3/2}+c$

15. $\frac{4}{27}(1+3x)3/2-\frac{4}{9}(1+3x)^{1/2}+c$

16. $\frac{2}{7}(1+x)^{7/2}-0.8(1+x)^{5/2}+c$

17. $-\frac{1}{6}(3x-1)^{2}+c$

18. $\frac{1}{256}(1+4x)^{16}-\frac{1}{240}(1+4x)^{15}+c$

19. $\frac{1}{14}(x+2)^{14}-\frac{5}{13}(x+2)^{13}+c$

20. $\sin x-\frac{1}{3}\sin^{3}x+c$

21. $-\cos x+\frac{2}{3}\cos^{3}x-0.2\cos^{5}x+c$

22. $\ln\left(1-e^{-x}\right)+c$

23. $\frac{1}{2}x+\frac{1}{4}\sin 2x+c$

24. $\frac{1}{2}x-\frac{1}{12}\sin 6x+c$

25. $\frac{1}{1+x}-\frac{1}{2+x}.\ln(1+x)-\ln(2+x)+c$

26. (a) $\ln(2+x)-\ln(3+x)+c$ (b) $2ln(2+x)-2\ln(1+x)-\frac{3}{1+x}+c$

(c) $\frac{1}{2}\ln(1+2x)-\frac{1}{3}(1-3x)+c$

16.2.2

1. $-xe^{-x}-e^{-x}+c$

2. $x\sin x+\cos x+c$

3. $-x\cos x+\sin x+c$

4. $\frac{1}{2}x^{2}\ln x-\frac{1}{4}x^{2}+c$

5. $\frac{1}{3}x^{3}\ln x-\frac{1}{9}x^{3}+c$

6. $\frac{x^{n+1}}{n+1}\ln x-\frac{x^{n+1}}{(n+1)^{2}}+c$

7. $x\sin^{-1}x+\sqrt{\left(1-x^{2}\right)}+c$

8. $x\tan^{-1}x-\frac{1}{2}\ln\left(1+x^{2}\right)+c$

9. $x^{2}e^{x}-2xe^{x}+2e^{x}+c$

10. $\frac{x^{2}e^{ax}}{a}-\frac{2xe^{ax}}{a^{2}}+\frac{2e^{ax}}{a^{3}}+c$

11. $\frac{1}{3}x^{2}\sin 3x+\frac{2}{9}\cos 3x-\frac{2}{27}\sin 3x+c$

12. $xe^{x}-4e^{x}+c$

13. $-\frac{x^{2}}{a}\cos ax+\frac{2x}{a^{2}}\sin ax+\frac{2}{a^{3}}\cos ax+c$

14. $-\frac{1}{x}\ln(x)-\frac{1}{x}+c$

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References:

Solomon, R.C. (1997), *A Level: Mathematics* (4th Edition) , Great Britain, Hillman Printers(Frome) Ltd.

More: (in Thai)

http://home.kku.ac.th/wattou/service/m456/09.pdf